

The Generalized Asymptotic Equipartition Property: Necessary and Sufficient Conditions

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Abstract

Suppose a string $X_1^n = (X_1, X_2, \dots, X_n)$ generated by a memoryless source $(X_n)_{n \geq 1}$ with distribution P is to be compressed with distortion no greater than $D \geq 0$, using a memoryless random codebook with distribution Q . The compression performance is determined by the “generalized asymptotic equipartition property” (AEP), which states that the probability of finding a D -close match between X_1^n and any given codeword Y_1^n , is approximately $2^{-nR(P,Q,D)}$, where the rate function $R(P, Q, D)$ can be expressed as an infimum of relative entropies. The main purpose here is to remove various restrictive assumptions on the validity of this result that have appeared in the recent literature. Necessary and sufficient conditions for the generalized AEP are provided in the general setting of abstract alphabets and unbounded distortion measures. All possible distortion levels $D \geq 0$ are considered; the source $(X_n)_{n \geq 1}$ can be stationary and ergodic; and the codebook distribution can have memory. Moreover, the behavior of the matching probability is precisely characterized, even when the generalized AEP is not valid. Natural characterizations of the rate function $R(P, Q, D)$ are established under equally general conditions.

Index Terms

Rate-distortion theory, data compression, large deviations, asymptotic equipartition property, random codebooks, pattern-matching

I. INTRODUCTION

Suppose a random string $X_1^n = (X_1, X_2, \dots, X_n)$ produced by a memoryless source $(X_n)_{n \geq 1}$ with distribution P on a source alphabet S , is to be compressed with distortion no more than some $D \geq 0$ with respect to a single-letter distortion measure $\rho(x, y)$.¹ The basic information-theoretic model for understanding the best performance that can be achieved, is the study of random codebooks. If we generate memoryless random strings $Y_1^n = (Y_1, Y_2, \dots, Y_n)$ according to some distribution Q on the reproduction alphabet T , we would like to know how many such strings are needed so that, with high probability, we will be able to find at least one codeword Y_1^n that matches the source string X_1^n with distortion D or less. The crucial mathematical problem in answering this question is the evaluation of the probability that a given, typical X_1^n , will be D -close to a random Y_1^n . This probability can be expressed as

$$\text{Prob}\{Y_1^n \in B_n(X_1^n, D) \mid X_1^n\} = Q^n(B_n(X_1^n, D)) \quad (1)$$

where $B_n(X_1^n, D)$ denotes the “distortion ball” consisting of all reproduction strings that are within distortion D (or less) from X_1^n ; note that the matching probability in (1) is itself a random quantity, as it depends on the source string X_1^n .

The importance of evaluating (1) was already identified by Shannon in his classic study of rate-distortion theory [15], where he showed that, for the best codebook distribution $Q = Q^*$, we have,

$$Q^{*n}(B_n(X_1^n, D)) \approx 2^{-nR(P,D)} \quad (2)$$

where $R(P, D)$ is the rate-distortion function of the source.

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¹Precise rigorous definitions are given in the following section.

The more general question of evaluating the matching probability (1) for distributions Q perhaps different from the optimal reproduction distribution Q^* , arises naturally in a variety of contexts, including problems in pattern-matching, mismatched codebooks, Lempel-Ziv compression, combinatorial optimization on random strings, and others; see, e.g., [20] [13] [18] [12] [19] [4] [17] [2] [16], and the review and references in [5]. In this case, Shannon's estimate (2) is replaced by the so-called “*generalized asymptotic equipartition property*” (or generalized AEP), which states that,

$$-\frac{1}{n} \log Q^n(B_n(X_1^n, D)) \rightarrow R(P, Q, D) \quad \text{a.s.} \quad (3)$$

where “a.s.” stands for “almost surely” and refers to the random string X_1^n . The rate function $R(P, Q, D)$ is defined in a way that closely resembles the rate-distortion function definition,

$$R(P, Q, D) := \inf_W H(W \| P \times Q)$$

where $H(\cdot \| \cdot)$ denotes the relative entropy, and the infimum is over all (bivariate) probability distributions of random variables (U, V) with values on S and T , respectively, such that U has distribution P and the expected distortion $E[\rho(U, V)] \leq D$. (For a broad introduction to the generalized AEP, its applications and refinements, see [5] and the references therein.)

The study of the rate function $R(P, Q, D)$ and its properties is an important step in understanding the generalized AEP. In terms of lossy data compression, it is not hard to see that $R(P, Q, D)$ is equal to the compression rate achieved by a (typically mismatched) random codebook with distribution Q . In view of this, it is not surprising that the rate-distortion function turns out to be *equal* to $R(P, Q^*, D)$, when the codebook distribution is chosen optimally,

$$R(P, D) = \inf_Q R(P, Q, D)$$

with the infimum being over all probability distributions Q on the reproduction alphabet T . Another important and useful observation made by various authors in the recent literature is that $R(P, Q, D)$ can alternatively be expressed as a convex dual.

Although much is known about the generalized AEP and about $R(P, Q, D)$ [5], all known results are established under certain restrictive conditions. In most cases the codebook distribution is required to be memoryless, and when it is not, it is assumed that the distortion measure is bounded. Moreover, only distortion levels in a certain range are considered, and the case when

$$D = D_{\min}(P, Q) := \inf\{D : R(P, Q, D) < \infty\}$$

is always excluded.

The main point of this paper is to remove these constraints, and to analyze which (if any) are essential for the validity of the generalized AEP. Our motivation is twofold. On one hand, unnecessarily stringent conditions make the theoretical picture incomplete. On the other, there are applications which naturally require more general statements. For example, in the study of universal lossy compression, where the source distribution is not known a priori, how can we assume that the distortion value chosen will be in the appropriate range and will not coincide with D_{\min} ? (Specific applications of the results in this paper to central problems in universal lossy data compression will be developed in subsequent work.) Similarly, the usual constraints on the distortion measure may fail to hold even for some basic distortion measures, like squared error distortion in the case of continuous alphabets. And the lack of information about the generalized AEP at $D = D_{\min}$ makes it difficult to draw tight correspondences between lossy and lossless compression, cf. [5].

Thus motivated, we give *necessary and sufficient conditions* for the generalized AEP in (3), and we precisely characterize the behavior of the matching probability in the pathological situations when the generalized AEP fails. Our results hold for *all* values of D , and they cover arbitrary abstract alphabets and distortion measures. We also allow the source to be stationary and ergodic, and the codebook distribution

to have memory. We similarly extend the characterization of the rate function $R(P, Q, D)$ to the same level of generality. We show that it can *always* be written as a convex dual, and that a minimizer W in the definition of $R(P, Q, D)$ always exists (unless, of course, the infimum is taken over the empty set).

Sections II and III contain the main results. Section IV contains generalizations to the case when the codebook distribution has memory. The bulk of the paper is devoted to proofs, which are collected in Section V. Our main mathematical tool is a generalized, one-sided version of the Gärtner-Ellis theorem from large deviations. It is stated and proved in Section V-C, and it may be of independent interest. Finally, the important special case when $D = D_{\min}$ is analyzed using results about the recurrence properties of random walks with stationary increments.

II. CHARACTERIZATION OF THE RATE FUNCTION

Let S be the source alphabet with its associated σ -algebra \mathcal{S} , let (T, \mathcal{T}) be the reproduction alphabet, and take $\rho : S \times T \mapsto [0, \infty)$ to be a distortion measure. We only assume that (S, \mathcal{S}) and (T, \mathcal{T}) are Borel spaces² and that ρ is $\sigma(\mathcal{S} \times \mathcal{T})$ -measurable. Henceforth, these σ -algebras and the various product σ -algebras derived from them are understood from the context. We use the abbreviations r.v., a.s., i.o., l.s.c., u.s.c. and log for random variable, almost surely, infinitely often, lower semicontinuous, upper semicontinuous and \log_e , respectively. If U and V are r.v.'s and $g(u) := Ef(u, V)$, we use the notation $E_V f(U, V)$ for the r.v. $g(U)$. When U and V are independent, then $E_V f(U, V) \stackrel{\text{a.s.}}{=} E[f(U, V)|U]$.

We write X and Y for two independent r.v.'s taking values in S and T , respectively, with $X \sim P$ and $Y \sim Q$. We use ρ to define a sequence of single-letter distortion measures ρ_n on $S^n \times T^n$, $n \geq 1$, by

$$\rho_n(x_1^n, y_1^n) := \frac{1}{n} \sum_{k=1}^n \rho(x_k, y_k)$$

where $x_i^j := (x_i, \dots, x_j)$. The dependence on ρ or ρ_n is suppressed in nearly all of our notation. We use

$$B_n(x_1^n, D) := \{y_1^n \in T^n : \rho_n(x_1^n, y_1^n) \leq D\}$$

to denote the distortion ball of radius D around x_1^n .

If W is a probability distribution on $S \times T$, then we use W_S to denote the marginal distribution of W on S , and similarly for W_T . An important subset of probability distributions on $S \times T$ is

$$W(P, D) := \{W : W_S = P, E_{(U,V) \sim W} \rho(U, V) \leq D\}.$$

This subset comes up in the definition of the rate-distortion function

$$R(P, D) := \inf_{W \in W(P, D)} H(W \| W_S \times W_T)$$

which we take to be $+\infty$ when $W(P, D)$ is empty. $H(\mu \| \nu)$ denotes the relative entropy (in nats).

$$H(\mu \| \nu) := \begin{cases} E_\mu \log \frac{d\mu}{d\nu} & \text{if } \mu \ll \nu, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $H(W \| W_S \times W_T)$ is the mutual information between r.v.'s (U, V) with joint distribution W .

Since $H(W \| W_S \times W_T) = \inf_Q H(W \| W_S \times Q)$, analysis of $R(P, D)$ often proceeds by expanding the infimum into two parts, namely,

$$\begin{aligned} R(P, D) &= \inf_Q R(P, Q, D) \\ R(P, Q, D) &:= \inf_{W \in W(P, D)} H(W \| P \times Q). \end{aligned}$$

²Borel spaces include \mathbb{R}^d as well as a large class of infinite-dimensional spaces, including Polish spaces. This assumption is made so that we can avoid certain pathologies while working with random sequences and conditional distributions [10].

The first infimum is over all probability distributions Q on T . Expanding the definition in this way is convenient, because $R(P, Q, D)$ can be expressed as a simple Fenchel-Legendre transform. In particular, define

$$\begin{aligned}\Lambda(P, Q, \lambda) &:= E_X [\log E_Y e^{\lambda \rho(X, Y)}] \\ \Lambda^*(P, Q, D) &:= \sup_{\lambda \leq 0} [\lambda D - \Lambda(P, Q, \lambda)].\end{aligned}$$

Proposition 1: $R(P, Q, D) = \Lambda^*(P, Q, D)$ for all D . If $W(P, D)$ is not empty, then this set contains a W such that $R(P, Q, D) = H(W \| P \times Q)$.

This alternative characterization is well known (see [5] for a review and references). We state it as a proposition and prove it below because typically it is qualified by other assumptions on ρ and D . In particular, the case $D = D_{\min}(P, Q)$ is almost always excluded, where

$$D_{\min}(P, Q) := \inf\{D : R(P, Q, D) < \infty\}.$$

$R(P, Q, D)$ has two other important characterizations that arise in a variety of contexts. Let $P_{x_1^n}$ denote the empirical distribution on S of x_1^n , let Q^n denote the n -times product measure of Q on T^n and define

$$L_n(x_1^n, Q_n, D) := -\frac{1}{n} \log Q_n(B_n(x_1^n, D))$$

for any probability distribution Q_n on T^n .

Theorem 2: If $(X_n)_{n \geq 1}$ is stationary and ergodic, taking values in S , with $X_1 \sim P$, then

$$\liminf_{n \rightarrow \infty} L_n(X_1^n, Q^n, D) \stackrel{\text{a.s.}}{=} R(P, Q, D)$$

for all D . The result also holds with $L_n(X_1^n, Q^n, D)$ replaced by $R(P_{X_1^n}, Q, D)$.

Of course, if the limit exists, then the \liminf is the also the limit and Theorem 2 is what Dembo and Kontoyiannis [5] call the *generalized AEP*. There are, however, pathological situations where the limit does not exist. In the next section we give necessary and sufficient conditions for the existence of the limit and we analyze in detail the situation where the limit does not exist.

III. THE GENERALIZED AEP

Here and in the remainder of the paper we will always assume that $(X_n)_{n \geq 1}$ is stationary and ergodic, taking values in S , with $X_1 \sim P$. Define³

$$\rho_Q(x) := \text{ess inf } \rho(x, Y).$$

We can exactly characterize when the \liminf is actually a limit in Theorem 2.

Theorem 3: $\lim_n L_n(X_1^n, Q^n, D)$ does not exist with positive probability if and only if $0 < D = D_{\min}(P, Q) < \infty$ and $R(P, Q, D) < \infty$ and $\rho_Q(X_1)$ is not a.s. constant. Furthermore, in this situation

$$\text{Prob}\{L_n(X_1^n, Q^n, D) = \infty \text{ i.o.}\} > 0 \tag{4a}$$

$$\text{Prob}\{L_n(X_1^n, Q^n, D) < \infty \text{ i.o.}\} = 1 \tag{4b}$$

$$\lim_{m \rightarrow \infty} L_{N_m}(X_1^{N_m}, Q^{N_m}, D) \stackrel{\text{a.s.}}{=} R(P, Q, D) \tag{4c}$$

where $(N_m)_{m \geq 1}$ is the (a.s.) infinite random subsequence of $(n)_{n \geq 1}$ for which $L_n(X_1^n, Q^n, D)$ is finite. All of the above also holds with $L_n(X_1^n, Q^n, D)$ replaced by $R(P_{X_1^n}, Q, D)$.

³ The essential infimum of a random variable η , is $\text{ess inf } \eta := \inf\{r : \text{Prob}\{\eta < r\} > 0\}$.

Combined with Theorem 2, this gives necessary and sufficient conditions for the generalized AEP. Both theorems are proven below. The proof shows that $(N_m)_{m \geq 1}$ can also be (a.s.) characterized as the random subsequence for which

$$\frac{1}{n} \sum_{k=1}^n \rho_Q(X_k) \leq D. \quad (5)$$

Note that $D_{\min}(P, Q) = E\rho_Q(X_1)$, whenever the former is finite.

A simple example that illustrates the pathology is the following: Let $(X_n)_{n \geq 1}$ be the sequence 1, 0, 1, 0, ... with probability 1/2 and the sequence 0, 1, 0, 1, ... with probability 1/2, namely, the binary, stationary, periodic Markov chain (which is ergodic). Let Q be the point mass at 0, let $\rho(x, y) := |x - y|$ and let $D = 1/2$. Note that $\rho_Q(X_1) = X_1$ is not constant, that $D = D_{\min}(P, Q) = 1/2$ and that $R(P, Q, D) = 0$ is finite. In the case when $X_1 = 0$, $L_n(X_1^n, Q^n, D) = 0$ for all n . In the case when $X_1 = 1$, however, $L_{2n}(X_1^{2n}, Q^{2n}, D) = 0$ and $L_{2n-1}(X_1^{2n-1}, Q^{2n-1}, D) = \infty$ for all n .

IV. EXTENSIONS TO THE CASE WITH MEMORY

Although the source $(X_n)_{n \geq 1}$ can have memory, the generalized AEP stated thus far is restricted to the case where the reproduction distribution is memoryless, that is, L_n is evaluated with a product measure Q^n . We relax this assumption here.

Let \mathbb{P} denote the distribution of $(X_n)_{n \geq 1}$, which we continue to assume is stationary and ergodic with $X_1 \sim P$. Let \mathbb{Q} denote the distribution of a stationary random process $(Y_n)_{n \geq 1}$ taking values in T with $Y_1 \sim Q$. We use P_n and Q_n to denote the distributions of X_1^n and Y_1^n , respectively, which are assumed to be independent. The results stated so far assume that \mathbb{Q} is memoryless, that is, $Q_n = Q^n$.

For the results in this section, however, we assume that \mathbb{Q} satisfies the following strong mixing condition:

$$C^{-1}\mathbb{Q}(A)\mathbb{Q}(B) \leq \mathbb{Q}(A \cap B) \leq C\mathbb{Q}(A)\mathbb{Q}(B)$$

for some fixed $1 \leq C < \infty$ and any $A \in \sigma(Y_1^n)$ and $B \in \sigma(Y_{n+1}^\infty)$ and any n . Notice that this implies ergodicity and includes the cases where \mathbb{Q} is memoryless ($C = 1$) and where \mathbb{Q} is a hidden Markov model (HMM) whose underlying Markov chain has a finite state space with all (strictly) positive transition probabilities. For the special case of a finite state Markov chain, a formula for $R_\infty(\mathbb{P}, \mathbb{Q}, D)$ not involving limits was identified in [18].

Following the definition of $R(P, Q, D)$, define

$$R_n(P_n, Q_n, D) := \frac{1}{n} \inf_{W_n \in W_n(P_n, D)} H(W_n \| P_n \times Q_n)$$

where $W_n(P_n, D)$ is the subset of probability distributions on $S^n \times T^n$ defined analogously to $W(P, D)$ except with ρ_n instead of ρ . Also, let $\delta_{x_1^n}$ be the probability distribution on S^n that assigns probability one to the sequence x_1^n .

Theorem 4: Theorems 2 and 3 remain valid when Q^n is replaced by Q_n , $R(P_{X_1^n}, Q, D)$ is replaced by $R_n(\delta_{X_1^n}, Q_n, D)$ and $R(P, Q, D)$ is replaced by $R_\infty(\mathbb{P}, \mathbb{Q}, D)$, where

$$R_\infty(\mathbb{P}, \mathbb{Q}, D) := \lim_{n \rightarrow \infty} R_n(P_n, Q_n, D).$$

The existence of the limit in the definition of $R_\infty(\mathbb{P}, \mathbb{Q}, D)$ is part of the result. Define

$$D_{\min}(\mathbb{P}, \mathbb{Q}) := \inf \{D : R_\infty(\mathbb{P}, \mathbb{Q}, D) < \infty\}.$$

Note that the mixing conditions here are strong enough to ensure that

$$D_{\min}(P, Q) = D_{\min}(\mathbb{P}, \mathbb{Q}) \quad (6)$$

and that

$$\text{ess inf } \rho_n(x_1^n, Y_1^n) = \frac{1}{n} \sum_{k=1}^n \rho_Q(x_k) \quad (7)$$

which is why the results for memory can still be in terms of $D_{\min}(P, Q)$ and ρ_Q . Extending Theorem 3 to situations where these do not hold seems difficult. The generalized AEP for \mathbb{Q} with memory can also be found in [2], [3], [5] under more general mixing conditions but for bounded distortion measure ρ and for $D \neq D_{\min}(\mathbb{P}, \mathbb{Q})$.

Define

$$\begin{aligned} \Lambda_n(P_n, Q_n, \lambda) &:= E_{X_1^n} [\log E_{Y_1^n} e^{\lambda \rho_n(X_1^n, Y_1^n)}] \\ \Lambda_n^*(P_n, Q_n, D) &:= \frac{1}{n} \sup_{\lambda \leq 0} [\lambda D - \Lambda_n(P_n, Q_n, \lambda)]. \end{aligned}$$

Proposition 1 immediately gives

$$R_n(P_n, Q_n, D) = \Lambda_n^*(P_n, Q_n, D)$$

so $R_\infty(\mathbb{P}, \mathbb{Q}, D)$ is the limit of a sequence of Fenchel-Legendre transforms. Analogous to the memoryless case, it can also be characterized directly as a Fenchel-Legendre transform.

Proposition 5: Define

$$\begin{aligned} \Lambda_\infty(\mathbb{P}, \mathbb{Q}, \lambda) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(P_n, Q_n, n\lambda) \\ \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) &:= \sup_{\lambda \leq 0} [\lambda D - \Lambda_\infty(\mathbb{P}, \mathbb{Q}, \lambda)]. \end{aligned}$$

Then $R_\infty(\mathbb{P}, \mathbb{Q}, D) = \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D)$.

The existence of the limit in the definition of $\Lambda_\infty(\mathbb{P}, \mathbb{Q}, \lambda)$ is part of the result. Occasionally it is more convenient to rewrite

$$\Lambda_n^*(P_n, Q_n, D) = \sup_{\lambda \leq 0} \left[\lambda D - \frac{1}{n} \Lambda_n(P_n, Q_n, n\lambda) \right]. \quad (8)$$

This form makes it easy to show that $R_n(P_n, Q_n, D) = R(P, Q, D)$ and that $R_n(\delta_{x_1^n}, Q^n, D) = R(P_{x_1^n}, Q, D)$, so that whenever \mathbb{Q} is memoryless, $R_\infty(\mathbb{P}, \mathbb{Q}, D) = R(P, Q, D)$ and all the results coincide.

V. PROOFS

The proofs occasionally refer to $D_{\text{ave}}(P, Q) := E\rho(X, Y)$ for independent $X \sim P$ and $Y \sim Q$.

A. Properties of Λ and Λ^* for arbitrary distortion measures

A common assumption in the literature is that ρ is either bounded or satisfies some moment conditions, such as $D_{\text{ave}}(P, Q) < \infty$. Since we do not assume these things here, we need to reverify many properties of Λ and Λ^* that can be found elsewhere under stronger conditions. These properties lead to the generalized AEP under the usual condition that $D \neq D_{\min}$. More detailed proofs, including measurability issues, can be found in a technical report that preceded this paper [8].

In this section we will use the assumptions and notation from Section II, however, we will suppress the dependence on P and Q whenever possible. In particular, we will think about $\Lambda(\lambda) := \Lambda(P, Q, \lambda)$ and $\Lambda^*(D) := \Lambda^*(P, Q, D)$ as functions of λ and D , respectively. It is also convenient to temporarily redefine

$$D_{\min} := \inf\{D : \Lambda^*(D) < \infty\}$$

until the end of this section where we prove Proposition 1. Proposition 1 shows that $\Lambda^*(D) = R(P, Q, D)$, so both definitions of D_{\min} are equivalent. Note that everything in this section applies equally well to Λ_n , Λ_n^* and R_n as defined in Section IV.

We begin with the following Lemma which comes mostly from [6][Lem. 2.2.5, Ex. 2.2.24]. See also [5], [19].

Lemma 6: [6] Let Z be a real-valued, nonnegative random variable. Define

$$\Gamma(\lambda) := \log Ee^{\lambda Z}.$$

Γ is nondecreasing and convex. Γ is finite, nonpositive and C^∞ on $(-\infty, 0)$ with

$$\lim_{\lambda \uparrow 0} \Gamma(\lambda) = \Gamma(0) = 0 \quad \text{and} \quad \Gamma'(\lambda) = \frac{EZ e^{\lambda Z}}{Ee^{\lambda Z}}, \quad \lambda < 0.$$

Γ' is finite, nonnegative and nondecreasing on $(-\infty, 0)$ with

$$\lim_{\lambda \downarrow -\infty} \Gamma'(\lambda) = \text{ess inf } Z \quad \text{and} \quad \lim_{\lambda \uparrow 0} \Gamma'(\lambda) = EZ.$$

If $\text{ess inf } Z < EZ$, then Γ is strictly convex on $(-\infty, 0)$.

Define $\Gamma(\lambda, x) := \log Ee^{\lambda \rho(x, Y)}$. For fixed x , we can apply Lemma 6 to the r.v. $Z := \rho(x, Y)$ to get several regularity properties of $\Gamma(\cdot, x)$. It turns out that these regularity properties are preserved by expectations, i.e., they continue to hold for $\Lambda(\lambda) = E\Gamma(\lambda, X)$. A sufficient condition is that Λ be finite on $(-\infty, 0]$. This replaces the typical moment conditions on ρ . Note that if $\Lambda^*(D)$ is finite for some D , i.e., if D_{\min} is finite, then this condition is trivially satisfied.

Lemma 7: Λ is nondecreasing and convex. Suppose Λ is finite on $(-\infty, 0]$. Then Λ is nonpositive and C^1 on $(-\infty, 0)$ with $\lim_{\lambda \uparrow 0} \Lambda(\lambda) = \Lambda(0) = 0$ and

$$\Lambda'(\lambda) = E_X \left[\frac{E_Y \rho(X, Y) e^{\lambda \rho(X, Y)}}{E_Y e^{\lambda \rho(X, Y)}} \right], \quad \lambda < 0.$$

Λ' is finite, nonnegative and nondecreasing on $(-\infty, 0)$ with

$$\lim_{\lambda \downarrow -\infty} \Lambda'(\lambda) = E\rho_Q(X) \quad \text{and} \quad \lim_{\lambda \uparrow 0} \Lambda'(\lambda) = D_{\text{ave}}.$$

If $E\rho_Q(X) < D_{\text{ave}}$, then Λ is strictly convex on $(-\infty, 0)$.

Proof: The statements about Λ are trivial. We will focus on the properties of Λ' which follow more or less immediately from the convexity of Λ and the differentiability of $\Gamma(\cdot, x)$. Let Λ'_- and Λ'_+ be the left hand and right hand derivatives of Λ , respectively, which are finite for $\lambda < 0$. The monotone convergence theorem immediately gives $\Lambda'_-(\lambda) = E\Gamma'(\lambda, X)$ for $\lambda < 0$. (The same argument can be used as $\lambda \uparrow 0$.) This shows that $\Gamma'(\lambda, X)$ has finite expectation and lets us use the dominated convergence theorem to get that $\Lambda'_+(\lambda) = E\Gamma'(\lambda, X)$. (The same argument can be used as $\lambda \downarrow -\infty$.) So the left and right hand derivatives of Λ are identical and have the given form. Recall that a differentiable, convex function has a continuous derivative. ■

These properties of Λ give the following well known properties of Λ^* , which we state without proof, except for (9). See [6][Lem. 2.2.5] and [14][Thm. 23.5, Cor. 23.5.1, Thm. 25.1].

Lemma 8: Λ^* is convex, l.s.c., nonnegative, nonincreasing and continuous from the right. $\Lambda^* \equiv \infty$ on $(-\infty, D_{\min})$ and $\Lambda^* \equiv 0$ on $[D_{\text{ave}}, \infty)$. If $D \leq D_{\text{ave}}$, then $\Lambda^*(D) = \sup_{\lambda \in \mathbb{R}} [\lambda D - \Lambda(\lambda)]$. If $D_{\min} < \infty$ (so that Lemma 7 applies), then $D_{\min} = E\rho_Q(X)$, Λ^* is finite and C^1 on (D_{\min}, ∞) and

$$\Lambda^*(D_{\min}) = E_X [-\log E_Y \mathbf{1}\{\rho(X, Y) = \rho_Q(X)\}]. \quad (9)$$

If further $D_{\min} < D_{\text{ave}}$, then Λ^* is strictly convex (and thus strictly decreasing) on $(D_{\min}, D_{\text{ave}})$ and for each $D \in (D_{\min}, D_{\text{ave}})$ there exists a unique $\lambda_D < 0$ such that $\Lambda^*(D) = \lambda_D D - \Lambda(\lambda_D)$.

Proof: We only prove (9). Define

$$\tilde{\rho}(x, y) := \max\{\rho(x, y) - \rho_Q(x), 0\}$$

so that $\tilde{\rho}$ is a valid distortion measure and so that

$$\rho(x, Y) \stackrel{\text{a.s.}}{=} \tilde{\rho}(x, Y) + \rho_Q(x).$$

Let $\tilde{\Lambda}$ be defined analogously to Λ , except with $\tilde{\rho}$ instead of ρ . We have $\Lambda(\lambda) = \tilde{\Lambda}(\lambda) + \lambda D_{\min}$ so that

$$\begin{aligned} \Lambda^*(D_{\min}) &= \sup_{\lambda \leq 0} \left[\lambda D_{\min} - \tilde{\Lambda}(\lambda) - \lambda D_{\min} \right] = \lim_{\lambda \downarrow -\infty} -\tilde{\Lambda}(\lambda) \\ &= \lim_{\lambda \downarrow -\infty} E_X \left[-\log E_Y e^{\lambda \tilde{\rho}(X, Y)} \right] \\ &= E_X \left[-\log E_Y \left(\lim_{\lambda \downarrow -\infty} e^{\lambda \tilde{\rho}(X, Y)} \right) \right] \\ &= E_X \left[-\log E_Y \mathbf{1}\{\tilde{\rho}(X, Y) = 0\} \right] \\ &= E_X \left[-\log E_Y \mathbf{1}\{\rho(X, Y) = \rho_Q(X)\} \right]. \end{aligned}$$

We moved the limit inside the expectations using first the monotone convergence theorem and then the dominated convergence theorem. \blacksquare

1) *Proposition 1:* Proposition 1 is an immediate consequence of the next two lemmas. The proofs follow [5][Thm. 2] with minor modifications. Note that Proposition 1 and Lemma 8 imply that $D_{\min} = E\rho_Q(X)$ whenever the former is finite.

Lemma 9: If $W \in W(P, D)$, then $H(W\|P \times Q) \geq \Lambda^*(D)$.

Proof: Let $\psi : T \mapsto (-\infty, 0]$ be measurable. Then [5]

$$H(\tilde{Q}\|Q) \geq E_{V \sim \tilde{Q}} \psi(V) - \log E e^{\psi(Y)}$$

for any probability measure \tilde{Q} on T . Applying the previous inequality with $\psi(y) := \lambda \rho(x, y)$, for $\lambda \leq 0$, gives

$$H(W(\cdot|x)\|Q) \geq \lambda E_{V \sim W(\cdot|x)} \rho(x, V) - \log E e^{\lambda \rho(x, Y)}$$

where $W(\cdot|x)$ denotes the regular conditional distribution of V given $U = x$ for $(U, V) \sim W$. Taking expectations w.r.t. U and noting that $W \in W(P, D)$ gives

$$H(W\|P \times Q) = E_{U \sim P} H(W(\cdot|U)\|Q) \geq \lambda D - \Lambda(\lambda).$$

Optimizing over $\lambda \leq 0$ completes the proof. \blacksquare

Lemma 10: If $\Lambda^*(D) < \infty$, then there exists a $W \in W(P, D)$ with $H(W\|P \times Q) = \Lambda^*(D)$.

Proof: The proof makes frequent use of Lemma 8. If $D \geq D_{\text{ave}}$, then $\Lambda^*(D) = 0$ and $W := P \times Q$ achieves the equality. If $D_{\min} < D < D_{\text{ave}}$, then W defined by

$$\frac{dW}{d(P \times Q)}(x, y) := \frac{e^{\lambda_D \rho(x, y)}}{E e^{\lambda_D \rho(x, Y)}}$$

achieves the equality [5], where λ_D is uniquely chosen so that $\Lambda^*(D) = \lambda_D D - \Lambda(\lambda_D)$.

Finally, if $D = D_{\min} = E\rho_Q(X)$, then define W by

$$\frac{dW}{d(P \times Q)}(x, y) := \frac{\mathbb{1}\{y \in A(x)\}}{E\mathbb{1}\{Y \in A(x)\}}$$

where $A(x) = \{y : \rho(x, y) = \rho_Q(x)\}$. Note that Lemma 8 shows that $\Lambda^*(D) = E_X [-\log E_Y \mathbb{1}\{Y \in A(X)\}]$ which we have assumed is finite, so the denominator is positive P -a.s. and W is well-defined. It is easy to see that $W \in W(P, D)$ and that

$$\begin{aligned} H(W \| P \times Q) &= E \left[\frac{dW}{d(P \times Q)}(X, Y) \log \frac{dW}{d(P \times Q)}(X, Y) \right] \\ &= E \left[\frac{\mathbb{1}\{Y \in A(X)\}}{E_Y [\mathbb{1}\{Y \in A(X)\}]} \log \mathbb{1}\{Y \in A(X)\} \right] \\ &\quad - E \left[\frac{\mathbb{1}\{Y \in A(X)\}}{E_Y [\mathbb{1}\{Y \in A(X)\}]} \log E_Y [\mathbb{1}\{Y \in A(X)\}] \right] \\ &= 0 - E_X [\log E_Y \mathbb{1}\{Y \in A(X)\}] = \Lambda^*(D) \end{aligned}$$

which completes the proof. ■

B. Extensions to memory

Here we prove Proposition 5 and the claims in the text following Theorem 4, including the existence of $R(\mathbb{P}, \mathbb{Q}, D)$, under the assumptions of Section IV. The stationarity and mixing properties of \mathbb{Q} give $Q^n \ll Q_n \ll Q^n$, which proves (7), and they give

$$\begin{aligned} &C^{-1} \int_{T^n} \int_{T^m} f(y_1^{n+m}) Q_m(dy_{n+1}^{n+m}) Q_n(dy_1^n) \\ &\leq \int_{T^{n+m}} f(y_1^{n+m}) Q_{n+m}(dy_1^{n+m}) \\ &\leq C \int_{T^n} \int_{T^m} f(y_1^{n+m}) Q_m(dy_{n+1}^{n+m}) Q_n(dy_1^n) \end{aligned} \tag{10}$$

for any function $f \geq 0$. We make use of this property repeatedly. Note that if f factors, i.e., if $f(y_1^{n+m}) = g(y_1^n)h(y_{n+1}^{n+m})$ for $g, h \geq 0$, then (10) becomes

$$C^{-1} E g(Y_1^n) E h(Y_1^m) \leq E f(Y_1^{n+m}) \leq C E g(Y_1^n) E h(Y_1^m). \tag{11}$$

This gives

$$\begin{aligned} &C^{-1} [E_{Y_1^n} e^{n\lambda\rho_n(x_1^n, Y_1^n)}] [E_{Y_1^m} e^{m\lambda\rho_m(x_{n+1}^{n+m}, Y_1^m)}] \\ &\leq E_{Y_1^{n+m}} e^{(n+m)\lambda\rho_{n+m}(x_1^{n+m}, Y_1^{n+m})} \\ &\leq C [E_{Y_1^n} e^{n\lambda\rho_n(x_1^n, Y_1^n)}] [E_{Y_1^m} e^{m\lambda\rho_m(x_{n+1}^{n+m}, Y_1^m)}] \end{aligned}$$

which implies that

$$\begin{aligned} &\Lambda_n(\delta_{x_1^n}, Q_n, n\lambda) + \Lambda_m(\delta_{x_{n+1}^{n+m}}, Q_m, m\lambda) - \log C \\ &\leq \Lambda_{n+m}(\delta_{x_1^{n+m}}, Q_{n+m}, (n+m)\lambda) \\ &\leq \Lambda_n(\delta_{x_1^n}, Q_n, n\lambda) + \Lambda_m(\delta_{x_{n+1}^{n+m}}, Q_m, m\lambda) + \log C. \end{aligned} \tag{12}$$

Replacing x_k with X_k and taking expected values gives

$$\begin{aligned}
& \Lambda_n(P_n, Q_n, n\lambda) + \Lambda_m(P_m, Q_m, m\lambda) - \log C \\
& \leq \Lambda_{n+m}(P_{n+m}, Q_{n+m}, (n+m)\lambda) \\
& \leq \Lambda_n(P_n, Q_n, n\lambda) + \Lambda_m(P_m, Q_m, m\lambda) + \log C.
\end{aligned} \tag{13}$$

This final result implies several things. First, it shows that if $\Lambda_n(P_n, Q_n, n\lambda)$ is finite (infinite) for some n , then it is finite (infinite) for all n . It also shows that the sequence $\Lambda_n(P_n, Q_n, n\lambda) + \log C$ is subadditive, so the limit in the definition of Λ_∞ exists. In particular [10][Lemma 10.21],

$$\begin{aligned}
\Lambda_\infty(\mathbb{P}, \mathbb{Q}, \lambda) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(P_n, Q_n, n\lambda) \\
&= \inf_{n \geq N} \frac{1}{n} [\Lambda_n(P_n, Q_n, n\lambda) + \log C]
\end{aligned}$$

for any $N \geq 0$. This gives

$$\begin{aligned}
& \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) \\
&= \sup_{\lambda \leq 0} \left[\lambda D - \inf_{n \geq N} \frac{1}{n} [\Lambda_n(P_n, Q_n, n\lambda) + \log C] \right] \\
&= \sup_{n \geq N} \left[\sup_{\lambda \leq 0} \left[\lambda D - \frac{1}{n} \Lambda_n(P_n, Q_n, n\lambda) \right] - \frac{\log C}{n} \right] \\
&= \sup_{n \geq N} \left[\Lambda_n^*(P_n, Q_n, D) - \frac{\log C}{n} \right].
\end{aligned}$$

The last equality follows from (8) which is easy to prove by moving the $1/n$ outside of the supremum and optimizing over $n\lambda$ instead of λ . Since we always have

$$\begin{aligned}
\Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) &= \sup_{\lambda \leq 0} \lim_{n \rightarrow \infty} \left[\lambda D - \frac{1}{n} \Lambda(P_n, Q_n, n\lambda) \right] \\
&\leq \liminf_{n \rightarrow \infty} \sup_{\lambda \leq 0} \left[\lambda D - \frac{1}{n} \Lambda(P_n, Q_n, n\lambda) \right] \\
&= \liminf_{n \rightarrow \infty} \Lambda_n^*(P_n, Q_n, D)
\end{aligned}$$

we have also shown that

$$\begin{aligned}
\Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) &= \lim_{n \rightarrow \infty} \Lambda_n^*(P_n, Q_n, D) \\
&= \lim_{n \rightarrow \infty} R_n(P_n, Q_n, D) := R(\mathbb{P}, \mathbb{Q}, D).
\end{aligned}$$

This completes the proof of Proposition 5 and shows that $R(\mathbb{P}, \mathbb{Q}, D)$ exists.

Lastly, (13) shows that

$$\Lambda(P, Q, \lambda) - \log C \leq \frac{1}{n} \Lambda_n(P_n, Q_n, n\lambda) \leq \Lambda(P, Q, \lambda) + \log C$$

so $\Lambda^*(P, Q, D) - \log C \leq \Lambda_n^*(P_n, Q_n, D) \leq \Lambda^*(P, Q, D) + \log C$. This gives (6).

C. A large deviations result

For appropriate values of D , the generalized AEP is essentially a large deviations result. The next lemma summarizes what we need. It is basically a corollary of the Gärtner-Ellis Theorem. Note that Λ and Λ^* are redefined in this section.

Lemma 11: Let $(Z_n)_{n \geq 1}$ be a sequence of nonnegative, real-valued random variables such that

$$\Lambda(\lambda) := \lim_{n \rightarrow \infty} \frac{1}{n} \log E e^{n\lambda Z_n} \text{ exists}$$

for all $\lambda \in \mathbb{R}$. Define $\Lambda^*(D) := \sup_{\lambda \leq 0} [\lambda D - \Lambda(\lambda)]$. Then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\} \leq -\Lambda^*(D)$$

for all D . Furthermore, if Λ^* is strictly convex on (a, b) , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\} = -\Lambda^*(D)$$

for all $D \in (a, b]$.

Proof: For any $\lambda \leq 0$, $\text{Prob}\{Z_n \leq D\} \leq E e^{n\lambda(Z_n - D)}$, so

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\} \\ \leq -\lambda D + \limsup_{n \rightarrow \infty} \frac{1}{n} \log E e^{n\lambda Z_n} = -[\lambda D - \Lambda(\lambda)]. \end{aligned}$$

Optimizing over $\lambda \leq 0$ gives the upper bound.

Suppose Λ^* is strictly convex on (a, b) . Since Λ^* is nonnegative and decreasing, Λ^* must be finite and positive on (a, b) . The finiteness implies that Λ is finite on $(-\infty, 0]$. We will first show that

$$\Lambda^*(D) = \sup_{\lambda \in \mathbb{R}} [\lambda D - \Lambda(\lambda)] \quad D \leq b. \quad (14)$$

It is easy to see that Λ is increasing and convex with $\Lambda(0) = 0$, so we can choose a $0 \leq D' \leq \infty$ with $\Lambda(\lambda) \geq \lambda D'$ for all $\lambda \in \mathbb{R}$. If $D' = \infty$, then $\Lambda(\lambda) = \infty$ for $\lambda > 0$ and (14) holds for all D . If D' is finite and $D \leq D'$, then $\lambda D - \Lambda(\lambda) \leq \lambda D' - \Lambda(\lambda) \leq 0$ for all $\lambda > 0$, so (14) holds for all $D \leq D'$. The same inequality gives $\Lambda^*(D') = 0$, so $b \leq D'$.

Now we will prove the lower bound. If Λ is finite in some neighborhood of zero, then the lemma follows immediately from the Gärtner-Ellis Theorem as stated in [7][Thm. V.6]. If this is not the case, then we need to slightly modify the sequence (Z_n) before applying the theorem.

Fix $D \in (a, b]$ and choose $0 < \epsilon < D - a$. Let $(\hat{Z}_n)_{n \geq 1}$ be a sequence of nonnegative, real-valued r.v.'s with distribution $\hat{P}_n(\cdot) := \text{Prob}\{\hat{Z}_n \in \cdot\}$ defined by

$$\frac{d\hat{P}_n}{dP_n}(z) := \frac{e^{-n\epsilon z}}{E e^{-n\epsilon Z_n}} \quad z \geq 0$$

where $P_n(\cdot) := \text{Prob}\{Z_n \in \cdot\}$. We have

$$\begin{aligned} \log \text{Prob}\{Z_n \leq D\} &\geq \log P_n((D - \epsilon, D)) \\ &= \log \int_{D-\epsilon}^D \frac{E e^{-n\epsilon Z_n}}{e^{-n\epsilon z}} \hat{P}_n(dz) \\ &\geq \log E e^{-n\epsilon Z_n} + n\epsilon(D - \epsilon) + \log \hat{P}_n((D - \epsilon, D)). \end{aligned}$$

Taking limits gives

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\} \\ & \geq \Lambda(-\epsilon) + \epsilon D - \epsilon^2 + \liminf_{n \rightarrow \infty} \frac{1}{n} \log \hat{P}_n((D - \epsilon, D)). \end{aligned} \quad (15)$$

We want to apply the Gärtner-Ellis Theorem to the sequence $(\hat{P}_n)_{n \geq 1}$. Note that

$$E e^{n\lambda \hat{Z}_n} = \int e^{n\lambda z} \frac{e^{-n\epsilon z}}{E e^{-n\epsilon Z_n}} P_n(dz) = \frac{E e^{n(\lambda - \epsilon)Z_n}}{E e^{-n\epsilon Z_n}}$$

so

$$\hat{\Lambda}(\lambda) := \lim_{n \rightarrow \infty} \frac{1}{n} \log E e^{n\lambda \hat{Z}_n} = \Lambda(\lambda - \epsilon) - \Lambda(-\epsilon)$$

exists and is finite for all $\lambda \leq \epsilon$. In particular, it is finite in a neighborhood of 0. Note also that

$$\begin{aligned} \hat{\Lambda}^*(x) &:= \sup_{\lambda \in \mathbb{R}} [\lambda x - \hat{\Lambda}(\lambda)] = \sup_{\lambda \in \mathbb{R}} [(\lambda + \epsilon)x - \hat{\Lambda}(\lambda + \epsilon)] \\ &= \sup_{\lambda \in \mathbb{R}} [\lambda x - \Lambda(\lambda)] + \epsilon x + \Lambda(-\epsilon) = \Lambda^*(x) + \epsilon x + \Lambda(-\epsilon) \end{aligned}$$

for any $x \leq b$. So $\hat{\Lambda}^*$ is also strictly convex on (a, b) and the slope of any supporting line to $\hat{\Lambda}^*$ at a point in (a, b) is strictly less than ϵ . In particular, the slope of such a point is in the interior of the domain where $\hat{\Lambda}$ is finite. So the assumptions of the Gärtner-Ellis Theorem are satisfied and

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \hat{P}_n((D - \epsilon, D)) &\geq - \inf_{x \in (D - \epsilon, D)} \hat{\Lambda}^*(x) \\ &= - \inf_{x \in (D - \epsilon, D)} [\Lambda^*(x) + \epsilon x + \Lambda(-\epsilon)] \\ &\geq - \inf_{x \in (D - \epsilon, D)} [\Lambda^*(x) + \epsilon D + \Lambda(-\epsilon)] \\ &= -\Lambda^*(D) - \epsilon D - \Lambda(-\epsilon). \end{aligned}$$

Combining this with (15) gives

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\} \geq -\Lambda^*(D) - \epsilon^2.$$

Since ϵ was arbitrary, this completes the proof. ■

Lemma 12: Let Z be a real-valued, nonnegative random variable. Define $\Lambda^*(D) := \sup_{\lambda \leq 0} [\lambda D - \log E e^{\lambda Z}]$. Then

$$\log \text{Prob}\{Z \leq D\} \leq -\Lambda^*(D)$$

with equality for $D \leq \text{ess inf } Z$. Furthermore, $\log \text{Prob}\{Z \leq D\}$ is finite if and only if $-\Lambda^*(D)$ is finite.

Proof: For any $\lambda \leq 0$, $\log \text{Prob}\{Z \leq D\} \leq \inf_{\text{a.s.}} [\lambda D - \log E e^{\lambda Z}]$. Optimizing over $\lambda \leq 0$ gives the first bound. Suppose $D \leq \text{ess inf } Z$ so that $Z - D \geq 0$. In this case

$$\begin{aligned} \text{Prob}\{Z \leq D\} &= \text{Prob}\{Z = D\} = \lim_{\lambda \rightarrow -\infty} E e^{\lambda(Z-D)} \\ &= \inf_{\lambda \leq 0} E e^{\lambda(Z-D)} \end{aligned}$$

and

$$\log \text{Prob}\{Z \leq D\} = \inf_{\lambda \leq 0} [\log E e^{\lambda Z} - \lambda D] = -\Lambda^*(D).$$

Of course, if $D > \text{ess inf } Z$, then $-\infty < \log \text{Prob}\{Z \leq D\} \leq -\Lambda^*(D) \leq 0$, and everything is finite. ■

Corollary 13: Lemma 11 holds if $n^{-1} \log \text{Prob}\{Z_n \leq D\}$ is replaced by $-\Lambda_n^*(D)$, where

$$\Lambda_n^*(D) := \frac{1}{n} \sup_{\lambda \leq 0} [\lambda D - \log E e^{\lambda Z_n}].$$

Proof: $-\Lambda_n^*(D) \leq -[n\lambda D - \log E e^{n\lambda Z_n}]/n$. Taking limits and optimizing over $\lambda \leq 0$ gives the upper bound

$$\limsup_{n \rightarrow \infty} -\Lambda_n^*(D) \leq -\Lambda^*(D).$$

Lemma 12 shows that

$$\liminf_{n \rightarrow \infty} -\Lambda_n^*(D) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Prob}\{Z_n \leq D\},$$

which gives the lower bound in the second part of Lemma 11. ■

D. The generalized AEP

Now we will prove the main theorems in the text. We focus on the more general setting with memory described in Section IV since this includes the memoryless situation as a special case. The main idea is to fix a typical realization $(x_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ and then analyze the behavior of the sequence of r.v.'s $(Z_n)_{n \geq 1}$, where

$$Z_n := \rho_n(x_1^n, Y_1^n) := \frac{1}{n} \sum_{k=1}^n \rho(x_k, Y_k) \quad (16)$$

and where $(Y_n)_{n \geq 1}$ has distribution \mathbb{Q} . Using this terminology,

$$L_n(x_1^n, Q_n, D) = -\frac{1}{n} \log \text{Prob}\{Z_n \leq D\}$$

and

$$\begin{aligned} R_n(\delta_{x_1^n}, Q_n, D) &= \Lambda_n^*(\delta_{x_1^n}, Q_n, D) \\ &:= \frac{1}{n} \sup_{\lambda \leq 0} [\lambda D - \log E e^{\lambda Z_n}]. \end{aligned}$$

The proof proceeds in several stages. Proposition 5 allows us to use $\Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D)$ instead of $R_\infty(\mathbb{P}, \mathbb{Q}, D)$. We first prove the lower bound

$$\liminf_{n \rightarrow \infty} L_n(X_1^n, Q_n, D) \stackrel{\text{a.s.}}{\geq} \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) \quad (17)$$

for all D . Then we prove the upper bound

$$\limsup_{n \rightarrow \infty} L_n(X_1^n, Q_n, D) \stackrel{\text{a.s.}}{\leq} \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) \quad (18)$$

separately for the cases $D < D_{\min}(P, Q)$, $D > D_{\text{ave}}(P, Q)$ and $D_{\min}(P, Q) < D \leq D_{\text{ave}}(P, Q)$. The case $D = D_{\min}(P, Q)$ can be pathological in certain situations. For these situations we characterize the pathology as described in Theorem 3 (extended to the situation with memory). Note that even in the pathological situation when the limit does not exist, there is a subsequence along which the upper bound in (18) holds. This gives Theorem 2 (extended to the situation with memory). Finally, Lemma 12 allows us to replace $L_n(X_1^n, Q_n, D)$ with $R_n(\delta_{x_1^n}, Q_n, D)$ along the lines of Corollary 13, even in the pathological situation.

1) *The lower bound:* (12) shows that we can apply the subadditive ergodic theorem [10][Theorem 10.22] to

$$\Lambda_n(\delta_{X_1^n}, Q_n, n\lambda) + \log C$$

for $\lambda \leq 0$ or to

$$-\Lambda_n(\delta_{X_1^n}, Q_n, n\lambda) + \log C$$

for $\lambda \geq 0$ (so that everything is bounded above by $\log C$) to get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(\delta_{X_1^n}, Q_n, n\lambda) \stackrel{\text{a.s.}}{=} \Lambda_\infty(\mathbb{P}, \mathbb{Q}, D). \quad (19)$$

The right side is a constant because the limit is shift-invariant and the source is ergodic. Since Λ_n is increasing in λ , we can choose the exceptional set independently of λ .

Choosing $(x_n)_{n \geq 1}$ so that (19) holds and defining $(Z_n)_{n \geq 1}$ as in (16) allows us to apply the first part of Lemma 11 to get the lower bound (17). Note that Corollary 13 gives the same lower bound for $R_n(\delta_{X_1^n}, Q_n, D)$.

2) *The upper bound when $D < D_{\min}$ or $D > D_{\text{ave}}$:* When $\Lambda^*(\mathbb{P}, \mathbb{Q}, D) = \infty$, the lower bound (17) implies the upper bound (18). Note that this includes all $D < D_{\min}(\mathbb{P}, \mathbb{Q})$ and possibly some situations where $D = D_{\min}(\mathbb{P}, \mathbb{Q})$.

If $D_{\text{ave}}(P, Q)$ is finite and $D > D_{\text{ave}}(P, Q)$, then Chebyshev's inequality and the ergodic theorem give

$$\begin{aligned} L_n(X_1^n, Q_n, D) &= -\frac{1}{n} \log [1 - Q_n \{y_1^n : \rho_n(X_1^n, y_1^n) > D\}] \\ &\leq -\frac{1}{n} \log \left[1 - \frac{1}{D} E_{Y_1^n} \rho_n(X_1^n, Y_1^n) \right] \\ &\stackrel{\text{a.s.}}{\rightarrow} 0 \leq \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) \end{aligned}$$

as $n \rightarrow \infty$, since $E_{Y_1^n} \rho_n(X_1^n, Y_1^n) \stackrel{\text{a.s.}}{\rightarrow} D_{\text{ave}}(P, Q) < D$. This gives the upper bound (18) for the case $D > D_{\text{ave}}(P, Q)$.

3) *The upper bound when $D_{\min} < D \leq D_{\text{ave}}$:* Assume that $D_{\min} := D_{\min}(\mathbb{P}, \mathbb{Q}) < D \leq D_{\text{ave}}(P, Q) := D_{\text{ave}}$. If $\Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, \cdot)$ is known to be strictly convex on $(D_{\min}, D_{\text{ave}})$, then we could apply the second part of Lemma 11 in the same manner as Section V-D1 to get the upper bound on $(D_{\min}, D_{\text{ave}}]$. Unfortunately, we were unable to find a simple proof of this strict convexity. Instead we will apply Lemma 11 to an approximating sequence of random variables $(\hat{Z}_n)_{n \geq 1}$.

Fix $m \in \mathbb{N}$. Let $\hat{\mathbb{Q}}$ denote the distribution of a random process $(\hat{Y}_n)_{n \geq 1}$ taking values in T with the property that $\hat{Y}_{(n-1)m+1}^{nm}$ has distribution Q_m and is independent of all the other \hat{Y}_k 's. We use \hat{Q}_n to denote the distribution of \hat{Y}_1^n . If $n = m\ell + r$, $1 \leq r \leq m$, then $\hat{Q}_n = (\times_{k=1}^\ell Q_m) \times Q_r$ and

$$C^{-\ell} \hat{Q}_n(A) \leq Q_n(A) \leq C^\ell \hat{Q}_n(A). \quad (20)$$

The next Lemma summarizes how $\hat{\mathbb{Q}}$ behaves in our context.

Lemma 14: Fix $m \in \mathbb{N}$ and define $\hat{\mathbb{Q}}$ as above. Then

$$\begin{aligned} \Lambda_\infty(\delta_{X_1^\infty}, \hat{\mathbb{Q}}, \lambda) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(\delta_{X_1^n}, \hat{Q}_n, n\lambda) \\ &= \frac{1}{m} \Lambda_m(P_m(\cdot|I), Q_m, m\lambda) \end{aligned} \quad (21)$$

exists and has the above representation for all $\lambda \in \mathbb{R}$ with probability 1, where $P_m(\cdot|I)$ is a random probability distribution on S^m depending only on the sequence X_1^∞ . Furthermore,

$$\Lambda_\infty^*(\delta_{X_1^\infty}, \hat{\mathbb{Q}}, D) := \sup_{\lambda \leq 0} \left[\lambda D - \Lambda_\infty(\delta_{X_1^\infty}, \hat{\mathbb{Q}}, \lambda) \right]$$

is strictly convex in D on $(D_{\min}, D_{\text{ave}})$ and

$$\begin{aligned}\Lambda_{\infty}^*(\delta_{X_1^{\infty}}, \hat{\mathbb{Q}}, D) - \frac{\log C}{m} &\leq \Lambda_{\infty}^*(\mathbb{P}, \mathbb{Q}, D) \\ &\leq \Lambda_{\infty}^*(\delta_{X_1^{\infty}}, \hat{\mathbb{Q}}, D) + \frac{\log C}{m}\end{aligned}\quad (22)$$

for all D with probability 1.

Proof: To simplify notation, fix λ and define the r.v.

$$\hat{\Lambda}_n := \Lambda(\delta_{X_1^n}, \hat{Q}_n, n\lambda).$$

We will first show that the convergence of $\hat{\Lambda}_n/n$ is a.s. determined by the convergence of the subsequence $\hat{\Lambda}_{m\ell}/(m\ell)$ as $\ell \rightarrow \infty$.

The ergodic theorem gives

$$\frac{1}{n} \sum_{k=1}^n \Lambda(\delta_{X_k}, Q, \lambda) \xrightarrow{\text{a.s.}} \Lambda(P, Q, \lambda). \quad (23)$$

Analogous to the arguments in Section V-B,

$$\frac{1}{n} \hat{\Lambda}_n \in \frac{1}{n} \sum_{k=1}^n \Lambda(\delta_{X_k}, Q, \lambda) \pm \log C. \quad (24)$$

If $\Lambda(P, Q, \lambda)$ is infinite, then (23) and (24) show that $\lim_n \hat{\Lambda}_n/n$ exists and is infinite a.s. In particular, $\lim_n \hat{\Lambda}_n/n \stackrel{\text{a.s.}}{=} \lim_{\ell} \hat{\Lambda}_{m\ell}/(m\ell)$.

If $\Lambda(P, Q, \lambda)$ is finite, then (23) shows that

$$\frac{1}{n} \Lambda(\delta_{X_n}, Q, \lambda) \xrightarrow{\text{a.s.}} 0$$

which implies that

$$\frac{1}{n} \Lambda_r(\delta_{X_{n-r+1}^n}, Q_r, r\lambda) \xrightarrow{\text{a.s.}} 0 \quad (25)$$

for each r ; see (12). Writing $n = m\ell + r$ for $1 \leq r \leq m$, the block-independence property of $\hat{\mathbb{Q}}$ gives

$$\hat{\Lambda}_n = \hat{\Lambda}_{m\ell} + \Lambda_r(\delta_{X_{\ell m+1}^n}, Q_r, r\lambda).$$

Combining this with (25) shows that $\hat{\Lambda}_n/n$ has a.s. the same asymptotic behavior as $\hat{\Lambda}_{m\ell}/(m\ell)$.

We will now analyze the limiting behavior of $\hat{\Lambda}_{m\ell}/(m\ell)$. The block-independence property of $\hat{\mathbb{Q}}$ gives

$$\frac{1}{m\ell} \hat{\Lambda}_{m\ell} = \frac{1}{m\ell} \sum_{k=1}^{\ell} \Lambda_m(\delta_{X_{m(k-1)+1}^{mk}}, Q_m, m\lambda). \quad (26)$$

The sequence $(X_{m(\ell-1)+1}^{m\ell})_{\ell \geq 1}$ of disjoint m -blocks from $(X_n)_{n \geq 1}$ is stationary (but not necessarily ergodic), so the ergodic theorem [10, Theorem 10.6] gives

$$\begin{aligned}\lim_{\ell \rightarrow \infty} \frac{1}{\ell} \sum_{k=1}^{\ell} \Lambda_m(\delta_{X_{(k-1)m+1}^{km}}, Q_m, m\lambda) \\ \stackrel{\text{a.s.}}{=} E [\Lambda_m(\delta_{X_1^m}, Q_m, m\lambda) | \mathcal{I}]\end{aligned}\quad (27)$$

where \mathcal{I} is the shift invariant σ -field for the sequence $(X_{m(\ell-1)m+1}^{m\ell})_{\ell \geq 1}$. Letting $P_m(\cdot | \mathcal{I})$ denote the regular conditional distribution of X_1^m given \mathcal{I} , the right side of (27) is $\Lambda_m(P_m(\cdot | \mathcal{I}), Q_m, m\lambda)$.

Combining (26) and (27) and recalling our discussion about the subsequence $(m\ell)_{\ell \geq 1}$ shows that (21) holds a.s. for each specific λ . Since Λ_n is increasing and since \mathcal{I} does not depend on λ , we can choose

the exceptional set independently of λ . This implies that the corresponding Λ_∞^* is a.s. well-defined and the exceptional set does not depend on D .

Two applications of the ergodic theorem show that

$$\begin{aligned}
D_{\text{ave}} &\stackrel{\text{a.s.}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n E_{Y_1} \rho(X_k, Y_1) \\
&= \lim_{\ell \rightarrow \infty} \frac{1}{m\ell} \sum_{k=1}^{\ell} \sum_{j=1}^m E_{Y_1} \rho(X_k, Y_1) \\
&= \frac{1}{\ell} \sum_{k=1}^{\ell} E_{Y_1^m} \rho_m(X_{(k-1)m+1}^{km}, Y_1^m) \\
&\stackrel{\text{a.s.}}{=} E [E_{Y_1^m} \rho_m(X_1^m, Y_1^m) | \mathcal{I}] \\
&= E_{X_1^m \sim P_m(\cdot | \mathcal{I})} [E_{Y_1^m} \rho_m(X_1^m, Y_1^m)].
\end{aligned} \tag{28}$$

An identical argument, combined with (7), gives

$$D_{\text{min}} \stackrel{\text{a.s.}}{=} E_{X_1^m \sim P_m(\cdot | \mathcal{I})} \left[\text{ess inf}_{Y_1^m} \rho_m(X_1^m, Y_1^m) \right]. \tag{29}$$

Because of the representation on the right side of (21), we can apply Lemma 8 with $S = S^m$, $T = T^m$, $\rho = \rho_m$, $X \sim P_m(\cdot | \mathcal{I})$, and $Y \sim Q_m$ to see that $\Lambda_\infty^*(\delta_{X_1^\infty}, \hat{\mathbb{Q}}, \cdot)$ is strictly convex on $(D_{\text{min}}, D_{\text{ave}})$ a.s. Identifying the D_{min} and D_{ave} from Lemma 8 with D_{min} and D_{ave} here follows from (29) and (28) above.

Finally, analogous to the arguments in Section V-B, (20) gives

$$\begin{aligned}
\Lambda_n(\delta_{x_1^n}, \hat{Q}_n, n\lambda) - \frac{n}{m} \log C &\leq \Lambda_n(\delta_{x_1^n}, Q_n, n\lambda) \\
&\leq \Lambda_n(\delta_{x_1^n}, \hat{Q}_n, n\lambda) + \frac{n}{m} \log C.
\end{aligned}$$

Combining this with (21) and (19) gives (22). ■

Returning to the main argument, fix a realization $(x_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ so that everything holds in Lemma 14. Define the sequence of random variables $(Z_n)_{n \geq 1}$ and $(\hat{Z}_n)_{n \geq 1}$ by $Z_n := \rho_n(x_1^n, Y_1^n)$ and $\hat{Z}_n := \rho_n(x_1^n, \hat{Y}_1^n)$. (20) shows that

$$\begin{aligned}
L_n(x_1^n, Q_n, D) &= -\frac{1}{n} \log Q_n(B_n(x_1^n, D)) \\
&\leq -\frac{1}{n} \log \hat{Q}_n(B_n(x_1^n, D)) + \frac{\log C}{m} \\
&= -\frac{1}{n} \log \text{Prob}\{\hat{Z}_n \leq D\} + \frac{\log C}{m}.
\end{aligned}$$

Lemma 14 lets us apply the second part of Lemma 11 to the right side to get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} L_n(x_1^n, Q_n, D) &\leq \Lambda_\infty^*(\delta_{X_1^\infty}, \hat{\mathbb{Q}}, D) + \frac{\log C}{m} \\
&\leq \Lambda_\infty^*(\mathbb{P}, \mathbb{Q}, D) + 2 \frac{\log C}{m}
\end{aligned}$$

for all $D \in (D_{\text{min}}, D_{\text{ave}}]$. The final inequality comes from (22). Since m was arbitrary and since $(x_n)_{n \geq 1}$ was a.s. arbitrary, we have established the upper bound (18) for the case $D_{\text{min}} < D \leq D_{\text{ave}}$.

4) *The case $D = D_{\min}$:* We have established the lower bound (17) for all D and the upper bound (18) for all D except for the case when $D = D_{\min} := D_{\min}(P, Q)$ and $\Lambda_{\infty}^*(\mathbb{P}, \mathbb{Q}, D_{\min}) < \infty$. We analyze that situation here. To simplify notation, we will suppress the dependence on \mathbb{P} and \mathbb{Q} whenever it is clear from the context.

Define

$$A_n(x_1^n) := \left\{ y_1^n \in T^n : \rho_n(x_1^n, y_1^n) = \operatorname{ess\,inf}_{Y_1^n} \rho_n(x_1^n, Y_1^n) \right\}.$$

Because of (7),

$$Q_{n+m}(A_{n+m}(x_1^{n+m})) = Q_{n+m}(A_n(x_1^n) \times A_m(x_{n+1}^{n+m}))$$

and the mixing properties of \mathbb{Q} give

$$\begin{aligned} & -\log Q_{n+m}(A_{n+m}(x_1^{n+m})) + \log C \\ & \leq [-\log Q_n(A_n(x_1^n)) + \log C] \\ & \quad + [-\log Q_m(A_m(x_{n+1}^{n+m})) + \log C]. \end{aligned}$$

Lemma 8 shows that

$$E[-\log Q_n(A_n(X_1^n))] = n\Lambda_{\infty}^*(P_n, Q_n, D_{\min})$$

which we assume is finite, so we can apply the subadditive ergodic theorem and Proposition 5 to get

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log Q_n(A_n(X_1^n)) \stackrel{\text{a.s.}}{=} \Lambda_{\infty}^*(\mathbb{P}, \mathbb{Q}, D_{\min}). \quad (30)$$

Note that if $\rho_Q(X_1)$ is a.s. constant, then $Q_n(A_n(X_1^n)) \stackrel{\text{a.s.}}{=} Q_n(B_n(X_1^n, D_{\min}))$ and (30) gives the upper bound.

Now suppose $\rho_Q(X_1)$ is not a.s. constant (and $D = D_{\min}$ and $\Lambda^*(D_{\min}) < \infty$). This is the only pathological situation where the upper bound does not hold. Our analysis makes use of recurrence properties for random walks with stationary and ergodic increments.⁴ What we need is summarized in the following lemma:

Lemma 15: Let $(U_n)_{n \geq 1}$ be a real-valued stationary and ergodic process and define $W_n := \sum_{k=1}^n U_k$, $n \geq 1$. If $EU_1 = 0$ and $\operatorname{Prob}\{U_1 \neq 0\} > 0$, then $\operatorname{Prob}\{W_n > 0 \text{ i.o.}\} > 0$ and $\operatorname{Prob}\{W_n \geq 0 \text{ i.o.}\} = 1$.

Proof: Define $W_0 := 0$. $(W_n)_{n \geq 0}$ is a random walk with stationary and ergodic increments. [11] shows that $\{\liminf_n n^{-1}W_n > 0\}$ and $\{W_n \rightarrow \infty\}$ differ by a null set. The ergodic theorem gives $\operatorname{Prob}\{n^{-1}W_n \rightarrow 0\} = 1$, so $\operatorname{Prob}\{W_n \rightarrow \infty\} = 0$. Similarly, by considering the process $-W_n$, we see that $\operatorname{Prob}\{W_n \rightarrow -\infty\} = 0$.

Now $\{|W_n| \rightarrow \infty\}$ is invariant and must have probability 0 or 1. If it has probability 1, then since we cannot have $W_n \rightarrow \infty$ or $W_n \rightarrow -\infty$ we must have W_n oscillating between increasingly larger positive and negative values, which means $\operatorname{Prob}\{W_n > 0 \text{ i.o.}\} = 1$ and completes the proof.

Suppose $\operatorname{Prob}\{|W_n| \rightarrow \infty\} = 0$. Define

$$N(A) := \sum_{n \geq 0} \mathbf{1}\{W_n \in A\}, \quad A \subset \mathbb{R},$$

to be the number of times the random walk visits the set A . [1][Corollary 2.3.4] shows that either $N(J) < \infty$ a.s. for all bounded intervals J or $\{N(J) = 0\} \cup \{N(J) = \infty\}$ has probability 1 for all intervals J (open or closed, bounded or unbounded, but not a single point). By assumption $|W_n| \not\rightarrow \infty$, so we can rule out the first possibility. Since $\operatorname{Prob}\{W_0 = 0\} = 1$, we see that for any interval J containing $\{0\}$ we must have $\operatorname{Prob}\{N(J) = \infty\} = 1$. In particular, taking $J := [0, \infty)$ shows that

⁴ $(W_n)_{n \geq 0}$ is a random walk with stationary and ergodic increments [1] if $W_0 := 0$ and $W_n := \sum_{k=1}^n U_k$, $n \geq 1$, for some stationary and ergodic sequence $(U_n)_{n \geq 1}$.

$\text{Prob}\{W_n \geq 0 \text{ i.o.}\} = 1$. Similarly, taking $J := (0, \infty)$ shows that $\text{Prob}\{W_n > 0 \text{ i.o.}\} = \text{Prob}\{N(J) = \infty\} = \text{Prob}\{N(J) > 0\} \geq \text{Prob}\{U_1 > 0\} > 0$. ■

Returning to the main argument,

$$\begin{aligned}
& L_n(X_1^n, Q_n, D_{\min}) \\
& \geq -\frac{1}{n} \log Q_n \left\{ y_1^n : \frac{1}{n} \sum_{k=1}^n \rho_Q(X_k) \leq D_{\min} \right\} \\
& = \begin{cases} 0 & \text{if } \sum_{k=1}^n \rho_Q(X_k) \leq n D_{\min} \\ \infty & \text{if } \sum_{k=1}^n \rho_Q(X_k) > n D_{\min} \end{cases} \\
& = \begin{cases} 0 & \text{if } W_n \leq 0 \\ \infty & \text{if } W_n > 0 \end{cases}, \tag{31}
\end{aligned}$$

where $W_n := \sum_{k=1}^n (\rho_Q(X_k) - D_{\min})$. Lemma 15 shows that $\text{Prob}\{W_n > 0 \text{ i.o.}\} > 0$. This and (31) prove (4a).

Lemma 15 also shows that $\text{Prob}\{W_n \leq 0 \text{ i.o.}\} = 1$. Let $(N_m)_{m \geq 1}$ be the (a.s.) infinite, random subsequence of $(n)_{n \geq 1}$ such that $W_n \leq 0$. Note that

$$\frac{1}{N_m} \sum_{k=1}^{N_m} \rho_Q(X_k) \leq D_{\min}$$

so

$$\begin{aligned}
& L_{N_m}(X_1^{N_m}, Q_{N_m}, D_{\min}) \\
& \leq -\frac{1}{N_m} \log Q_{N_m} \left(B_{N_m}(X_1^{N_m}, \frac{1}{N_m} \sum_{k=1}^{N_m} \rho_Q(X_k)) \right) \\
& = -\frac{1}{N_m} \log Q_{N_m}(A_{N_m}(X_1^{N_m})). \tag{32}
\end{aligned}$$

Now, the final expression in (32) is a.s. finite because $E[-\log Q_n(A_n(X_1^n))] = n\Lambda_n^*(D_{\min}) < \infty$. This proves (4b) and shows that $(N_m)_{m \geq 1}$ satisfies the claims of the theorem, including (5). Letting $m \rightarrow \infty$ in (32) and using (30) gives (4c), the upper bound along the sequence $(N_m)_{m \geq 1}$. Note that it also shows that the \liminf_n is a.s. Λ_∞^* even in this pathological case.

5) *Replacing L_n with R_n :* Defining $Z_n := \rho_n(x_1^n, Y_1^n)$, Proposition 1 and Lemma 12 show that

$$R_n(\delta_{x_1^n}, Q_n, D) = \Lambda_n^*(\delta_{x_1^n}, Q_n, D) \leq L_n(x_1^\infty, Q_n, D)$$

and that R_n and L_n are finite (infinite) together. Since we have already established that $L_n(X_1^\infty, Q_n, D)$ and $R_n(\delta_{X_1^n}, Q_n, D)$ have the same lower bound (17), we can use the above bound to squeeze R_n when ever $\lim_n L_n$ exists.

In the only pathological situation where the limit does not exist, L_n converges along the subsequence where it is finite, so R_n converges along that subsequence also. But as we noted above, L_n and R_n have the same subsequence where they are finite.

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